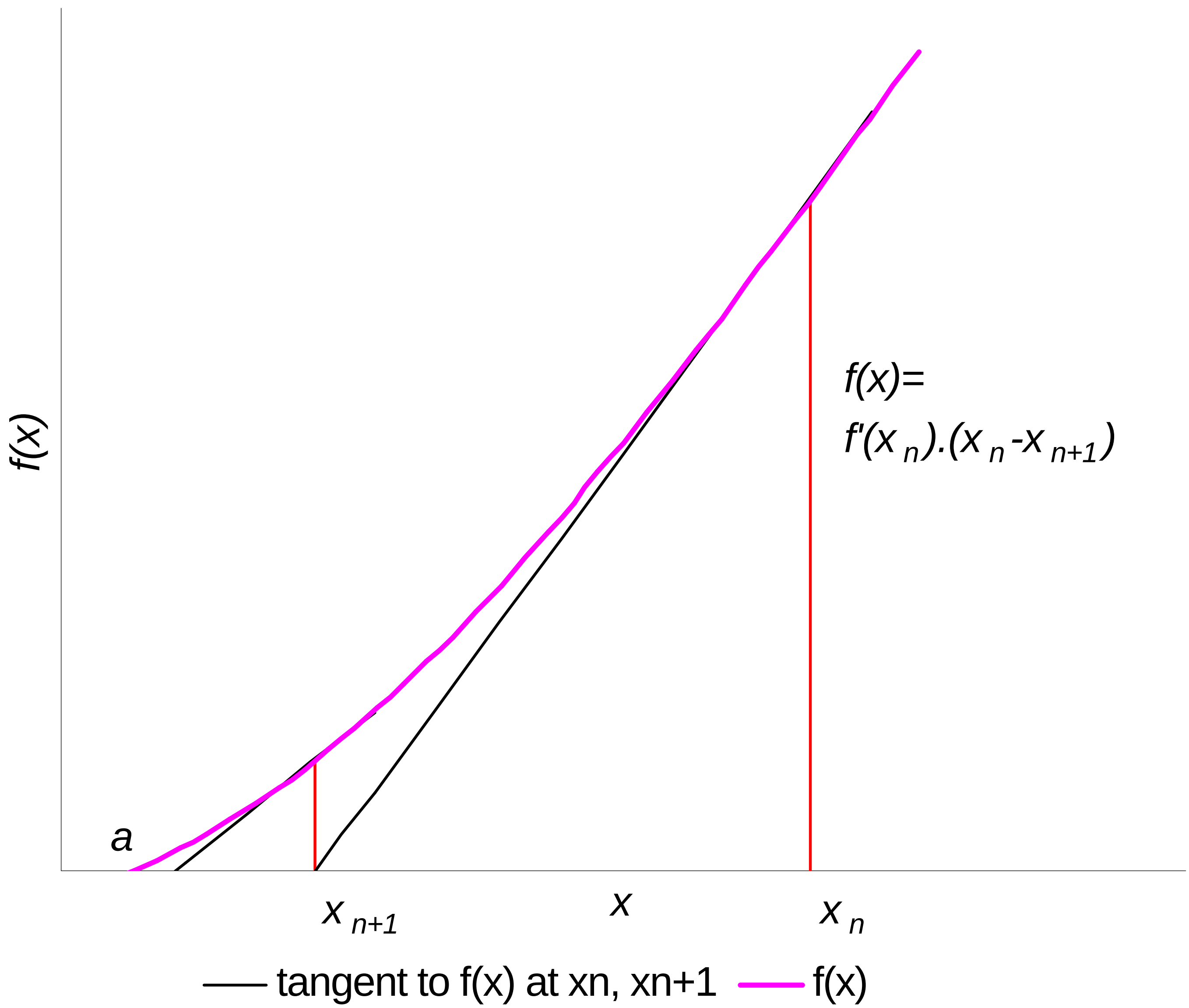
NEWTON-RAPHSON METHOD, also known as NEWTON'S METHOD FOR $f(x)=0$



Draw a tangent to the function at x_n. A better estimate x_{n+1} is found where the tangent cuts the x-axis.

From the (large) triangle with vertices at $x_{n+1},0 \;/\; x_n,0 \;/\; x_n, f(x_n)$,

we see that $f(x_n)=f'(x_n)\cdot(x_n-x_{n-1})$ i.e.

$$x_{n+1}:=x_n-\frac{f(x_n)}{f'(x_n)}$$

PROOF OF CONVERGENCE OF NEWTON'S METHOD: BASIC ITERATION WITH $g'(x) = 0$

$$
\begin{aligned}
f(x) &= 0 \\
-\frac{f(x)}{f'(x)} &= 0 \qquad \textit{if} \;\; f'(x) \neq 0 \\
x - \frac{f(x)}{f'(x)} &= x \\
\textit{i.e.} \quad g(x) &= x \quad \textit{basic iteration}
\end{aligned}
$$

From above

$$g(x) \equiv x - \frac{f(x)}{f'(x)}$$

Evaluate $g'(x)$**:**

$$g'(x) \equiv 1 - \frac{\left(f'(x)\cdot f'(x) - f(x)\cdot f''(x)\right)}{f'(x)^2}$$

$$g'(x) \equiv 1 - 1 + \frac{f(x)\cdot f''(x)}{f'(x)^2}$$

At $x = a$, $f(x) = 0 \qquad \therefore g(x = a) = 0$.

This leaves convergence to the second order terms.

Convergence rates:

Method	*corrections*	*order*	*works with*
Basic iteration	$e_{n+1} \approx g'(x_n)e_n$	1^{st}	$\lvert g'(a)\rvert \leq 1$
Newton's method	$e_{n+1} \approx \frac{f''(x_n)e_n^2}{2f'(x_n)}$	2^{nd}	steep slope, low curvature

EXAMPLE 13: $f(x) = x - \cos x$

Newton's method provides the iterative scheme:

$$x_{n+1} = x_n - \frac{(x_n - \cos x_n)}{(1 + \sin x_n)}$$

$x_0 = 1$ $\quad x_1 = 1 \quad -\frac{0.46}{1.84} \quad$ =0.75

$r_1 \equiv x_1 - x_0 = 0.25$ $\quad e_1 \equiv x_1 - a = 0.01092$

$x_1 = 0.75$ $\quad x_2 = 0.75 \quad -\frac{0.01831}{1.682} \quad$ =0.73911

$r_2 = 0.011$ $\quad e_2 = 0.00003$

$x_2 = 0.73911$ $\quad x_3 = 0.73911 \quad -\frac{0.000048}{1.674} \quad$ =0.73908

$r_3 = 0.00003$ $\quad e_3 \simeq 0$

The solution has converged to 4DP and 4SF after the second step.

The method is very effective. The disadvantage is the need to evaluate the derivative $f'(x)$. This may be either tedious or impossible; in which case the *backward difference* form of $f'(x)$ is used:

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

So that the conventional form of Newton's method

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

becomes

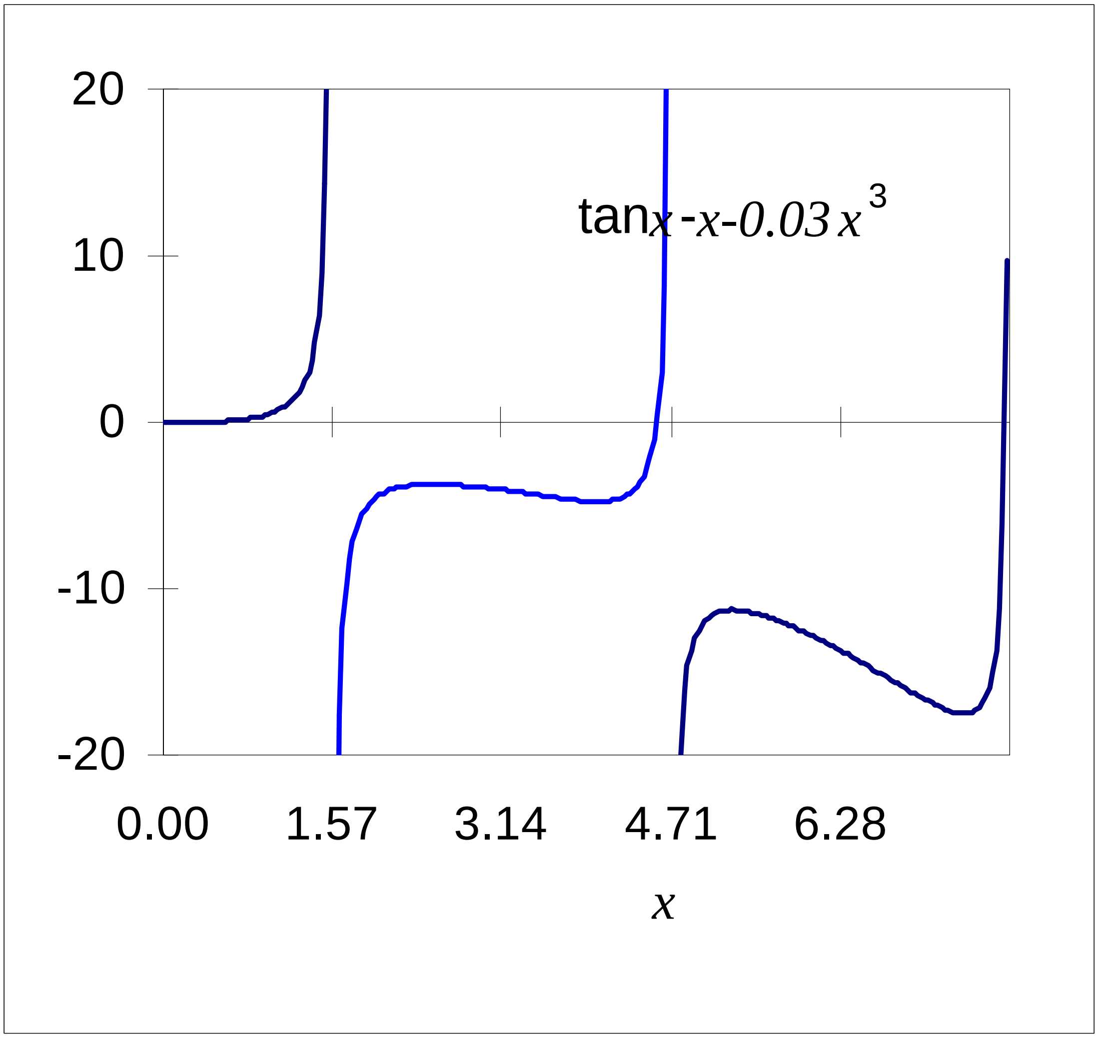
$$x_{n+1} := x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

This resembles Linear Interpolation except that extrapolation is possible and each limit is used for two steps only before replacement.

FAILURE OF NEWTON'S METHOD

In rare cases Newton's method won't work or converges to the wrong root.

EXAMPLE 14: $\tan x - x - 0.03x^3$



There is a root at 4.579 in the middle portion of the function. If we take 4.0 as initial estimate x_0, the iteration sequence proceeds;

$x_0 = 4$

$x_1 = -43.885$

$x_2 = -29$

$x_3 = -18.86$

$x_4 = -11.98$

$x_5 = -6.8$

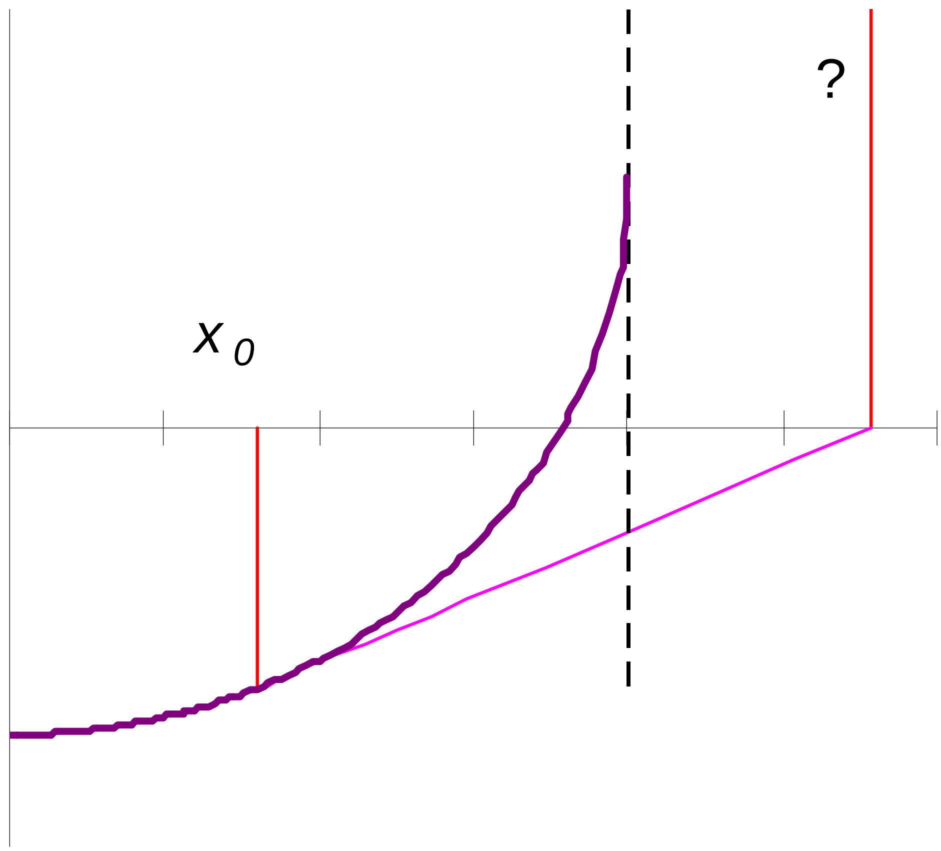
and eventually converges to x=0, the trivial root.

Starting with any $x_0 < 4.45$ also cannot converge to the correct root.

The steep curvature and multiple branches cause problems for the method.

EXAMPLE 15: $y = 0.9 - \sqrt{4-(x-1)^2}$

In this case (a quadrant of a circle) the extrapolated tangent at x_0 intercepts the *x*-axis beyond the asymptote so the root cannot be found.



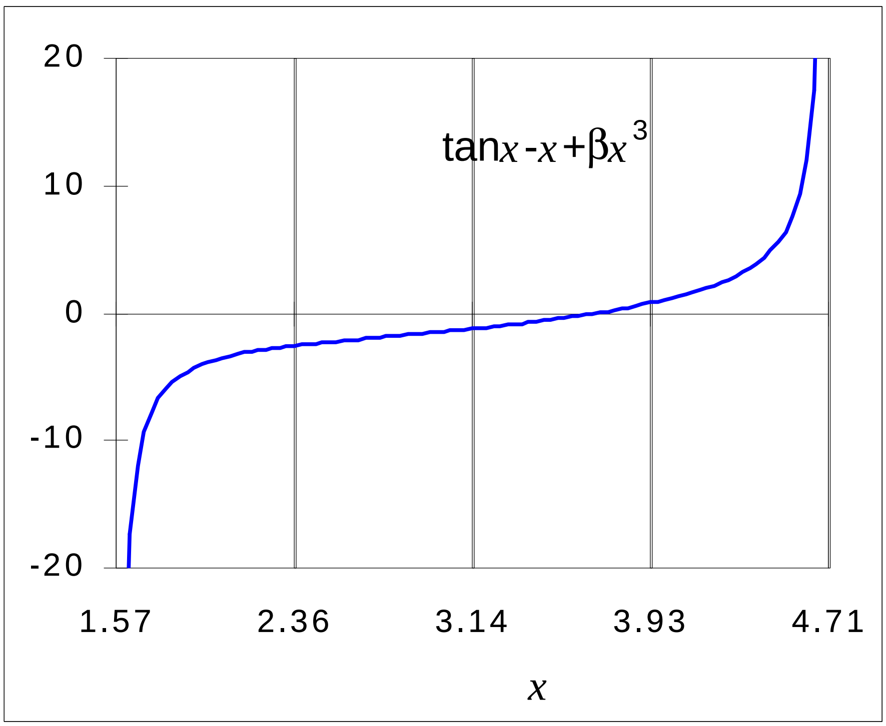
In both examples a better choice of initial estimate, based on sketching would lead to successful identification of the root.

<u>EXAMPLE 16:</u> ROOTS OF $\tan x - x\left(1-0.061x^2\right)$ in range $\pi/2, 3\pi/2$

First sketch the function:

x	$\tan x$	$0.061x^3$	$f(x)$
$\pi/2^+$	$\pm\infty$	0.798	$-\infty$
$3\pi/4$	1	1.892	-2.553
π	0	3.694	-1.238
$5\pi/4$	1	4.712	0.792
$3\pi/2^-$	$\pm\infty$	5.498	∞

sign change between $\pi, 5\pi/4$



From the plot (with a few added points!) we guess a root at 1.15π .

Use
$$f(x) = \tan x - x\left(1-0.061x^2\right)$$
$$f'(x) = \sec^2 x - 1 + 0.183x^2 \text{:}$$

Step 1

$x_0 = 3.6$

$f(x_0) = -0.24$ $\qquad$ $x_1 = 3.6 + \dfrac{0.24}{2.632}$ =3.691

$f'(x_0) = 2.632$

Step 2

$$x_1 = 3.691$$

$$f(x_1) = 0.0108 \qquad x_2 = 3.691 - \frac{0.0108}{2.886} \qquad = 3.687$$

$$f'(x_1) = 2.886$$

Step 3

$$x_2 = 3.687$$

$$f(x_2) = -8\times10^{-4} \qquad x_3 = 3.687 + \frac{8\times10^{-4}}{2.886} \qquad = 3.687$$

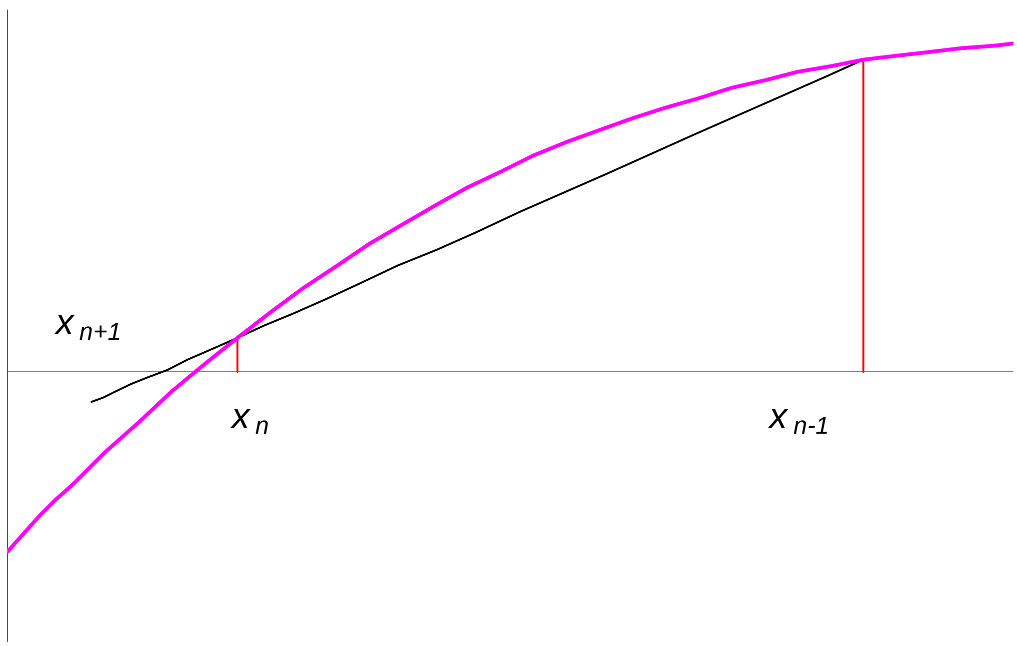
$$f'(x_2) \simeq f'(x_1) = 2.886$$

no change in 4th decimal.

SECANT METHOD

As suggested earlier, the derivative $f'(x)$ is replaced by the numerical backward difference form to give the algorithm:

$$x_{n+1} := x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$



The disadvantage is that two starting values (bounds on root obtained from sketching?) are needed.
However, for every step *only one* new function value is required, not two since the derivative is obtained numerically.

Secant method is most useful when function derivative is not known or is difficult or impossible to evaluate analytically e.g.

$$f(x) = \left(\ln\left(\left|\sin\sqrt{1-x^2}\right|\right)\cdot e^{3\cos\sqrt{x}}\right)^{1.5}$$

$f(l) = |\mathrm{A} - l\,\mathrm{I}|$ where A is large order matrix.

EXAMPLE 17: $f(x) = x - \cos x$

$x_0 = 1.0$ $\quad$ $x_1 = 0.9$

$f(x_0) = 0.46$ $\quad$ $f(x_1) = 0.278$

Apply secant formula –**step 1**

$$x_2 = x_1 - f(x_1) \times \frac{x_1 - x_0}{f(x_1) - f(x_0)} \qquad x_2 \simeq x_1 - f(x_1) \times \frac{1}{f'(x_1)}$$

$$x_2 = 0.9 - 0.278 \times \frac{0.9 - 1.0}{0.278 - 0.46} \qquad x_2 = 0.9 - 0.278 \times \frac{1}{1.82}$$

=0.747

$x_1 = 0.9$ $\quad$ $x_2 = 0.747$

$f(x_1) = 0.278$ $\quad$ $f(x_2) = 0.013$ $\quad$ new function evaluation

Apply secant formula –**step 2**

$$x_3 = x_2 - f(x_2) \times \frac{x_2 - x_1}{f(x_2) - f(x_1)} \qquad x_3 \simeq x_2 - f(x_2) \times \frac{1}{f'(x_2)}$$

$$x_3 = 0.747 - 0.013 \times \frac{0.747 - 0.90}{0.013 - 0.278} \qquad x_3 = 0.747 - 0.013 \times \frac{1}{1.73}$$

=0.7395

The changes are now quite small. For last step assume $f'(x_3) \simeq f'(x_2)$, and evaluate $f(x_3) = 0.0007$.

Apply secant formula –**step 3**

Hence $x_4 = 0.7395 - 0.0007 \times \frac{1}{1.73}$ =0.7391

Convergence is fast since f' is large and f'' is small.

SUMMARY FOR NEWTON'S METHOD

- Most effective method, equivalent to optimal form of basic iteration with $g'(x) = 0$.
- Can fail unless initial estimate is wisely chosen
- Secant method is useful when $f'(x)$ is difficult or impossible to evaluate. Method uses only one function evaluation per step and is slightly slower.
- Convergence is second order and is best for low curvature and steep gradient.
- Ideally suited for multi-variable problems

SUMMARY FOR ROOT FINDING

- Sketching with a few points is recommended to estimate roots and identify possible problems related to slope, curvature or asymptotes.
- Interval halving is slow but guaranteed to work
- Linear interpolation is faster (usually).
- Basic iteration can be very simple but depends on identifying best *g(x)*.
- Newton-Raphson Method (aka Newton's Method) is the best.
- Secant Method is used when analytical derivatives are not available.
- For more than one variable Newton's method is highly effective.
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